

DETERMINANT LINES, VON NEUMANN ALGEBRAS AND L^2 TORSION

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ABSTRACT. In this paper, we suggest a construction of determinant lines of finitely generated Hilbertian modules over finite von Neumann algebras. Nonzero elements of the determinant lines can be viewed as volume forms on the Hilbertian modules. Using this, we study both L^2 combinatorial and L^2 analytic torsion invariants associated to flat Hilbertian bundles over compact polyhedra and manifolds; we view them as volume forms on the reduced L^2 homology and L^2 cohomology. These torsion invariants specialize to the classical Reidemeister-Franz torsion and the Ray-Singer torsion in the finite dimensional case. Under the assumption that the L^2 homology vanishes, the determinant line can be canonically identified with \mathbb{R} , and our L^2 torsion invariants specialize to the L^2 torsion invariants previously constructed by A.Carey, V.Mathai and J.Lott. We also show that a recent theorem of Burghlea et al. can be reformulated as stating equality between two volume forms (the combinatorial and the analytic) on the reduced L^2 cohomology.

§0. Introduction

The study of the L^2 Reidemeister-Franz torsion and the L^2 analytic torsion was initiated in [M], [L] and [CM]. These invariants were originally defined for manifolds with trivial L^2 cohomology and positive Novikov-Shubin invariants; they were shown to be piecewise linear and smooth invariants respectively. See also [LR] for K-theoretic generalizations of these invariants.

In this paper, we introduce a new concept of determinant line of a finitely generated Hilbertian module over a von Neumann algebra. Here a Hilbertian module is defined as a topological vector space and a module over a von Neumann algebra such that there is an admissible scalar product on it, making it a Hilbert module. The construction of the determinant line, which we suggest here, generalizes the classical construction of determinant line of a finite dimensional vector space, and enjoys similar functorial properties. Nonzero elements of the determinant line can be naturally viewed as volume forms on the Hilbertian module. This enables us to make sense of the notions of volume forms and determinant lines in the infinite dimensional and non-commutative situation.

We then define analytic and the Reidemeister-Franz L^2 torsion invariants of flat Hilbertian bundles of determinant class over finite polyhedra and compact manifolds respectively. These reduce to the classical constructions in the finite dimensional situation. These new torsion invariants live in the determinant lines of reduced L^2

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cohomology. We prove that the combinatorial L^2 torsion is combinatorially invariant (i.e. it is invariant under subdivisions). Under the hypothesis that the L^2 homology vanishes, these determinant lines can be canonically identified with \mathbb{R} , and our L^2 torsion invariants reduce to the ones previously studied in [M], [L] and [CM].

In a recent preprint [BFKM] of Burghelea, Friedlander, Kappeler and McDonald, an equality between some combinatorial and analytic L^2 torsion invariants, was established. More precisely, in [BFKM] they introduced three numerical invariants T_{an} , T_{met} and T_{comb} such that T_{an} depends on the Riemannian metric, T_{comb} depends on the triangulation and T_{met} depends both on the Riemannian metric and the triangulation. The main result of [BFKM] states the equality $T_{an} = T_{comb} \times T_{met}$.

In this paper we observe (following the suggestion of the referee) that one can reformulate the the main theorem of [BFKM] as stating equality between the combinatorial and analytic torsion invariants as defined in this paper, i.e. understood as volume forms on the reduced L^2 cohomology.

The paper is organized as follows. In the first section, which contains preliminary material, we discuss basic properties of Hilbertian modules over von Neumann algebras, the canonical trace on the commutant and present a new construction of the Fuglede-Kadison determinant. This construction avoids reference to the scalar product (unlike the standard construction) and so it is well suited for Hilbertian modules which is crucial for this paper. In section §2, we construct determinant lines for finitely generated Hilbertian modules and establish their basic functorial properties. In section §3, we consider a generalisation, which enables to deal with D -admissible scalar products, D -isomorphisms etc.. Here the letter " D " stands for "*determinant class type*" condition. In section §4, we define combinatorial L^2 torsion, as an element in the determinant line of the reduced L^2 homology and cohomology; we also prove combinatorial invariance. In section §5, we define L^2 analytic torsion as an element in the determinant line of reduced L^2 cohomology. In last section §6, we show how one may reformulate the theorem of [BFKM] using the notions introduced in the present paper.

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§1. Hilbertian modules and the Fuglede-Kadison determinant

This section contains some preliminary material which will be used later in this paper.

First, we describe the notion of a *Hilbertian module* over a finite von Neumann algebra, which is distinct from the standard well-known notion of a *Hilbert module*. The difference is that Hilbertian module does not have a specified scalar product on it, and any choice of such scalar product introduces an additional structure. This notion will be crucial for our goals in the following sections. Secondly, we show that the trace on the initial von Neumann algebra determines canonically a trace on the commutant of any finitely generated Hilbertian module. At last, we use this canonical trace to define the Fuglede-Kadison (FK) determinant. Note, that the commutant of a Hilbertian module is not a von Neumann algebra (since it has no fixed $*$ -operator). In order to be able to use the standard theory of FK determinants, we have to show that it is independent on the involution (which is used in its definition, cf. [FK]). Instead, we provide in this section a very simple self-contained exposition of the FK

determinants, adjusted to needs of the current situation.

1.1. Let \mathcal{A} be a finite von Neumann algebra with a fixed finite, normal, and faithful trace $\tau : \mathcal{A} \rightarrow \mathbb{C}$. The involution in \mathcal{A} will be denoted $*$; by $\ell^2(\mathcal{A}) = \ell_\tau^2(\mathcal{A})$ we denote the completion of \mathcal{A} with respect to the scalar product $\langle a, b \rangle = \tau(b^*a)$, for $a, b \in \mathcal{A}$, determined by the trace τ .

1.2. Recall that a *Hilbert module* over \mathcal{A} is a Hilbert space M together with a continuous left \mathcal{A} -module structure such that there exists an isometric \mathcal{A} -linear embedding of M into $\ell^2(\mathcal{A}) \otimes H$, for some Hilbert space H . Note that this embedding is not part of the structure. A Hilbert module M is said to be *finitely generated* if it admits an embedding $M \rightarrow \ell^2(\mathcal{A}) \otimes H$ as above with finite dimensional H .

Any Hilbert module, being a Hilbert space, has a particular scalar product. In this paper we wish to consider a weaker notion obtained from Hilbert module by forgetting the scalar product but preserving its topology and the \mathcal{A} -action.

1.3. Definition. A *Hilbertian module* is a topological vector space M with continuous left \mathcal{A} -action such that there exists a scalar product $\langle \cdot, \cdot \rangle$ on M which generates the topology of M and such that M together with $\langle \cdot, \cdot \rangle$ and with the \mathcal{A} -action is a Hilbert module.

If M is a Hilbertian module, then any scalar product $\langle \cdot, \cdot \rangle$ on M with the above properties will be called *admissible*.

A *morphism* of Hilbertian modules $f : M \rightarrow N$ is a continuous linear map commuting with the \mathcal{A} -action.

Note that the kernel of any morphism f is again a Hilbertian module. Also, the closure of the image $\text{cl}(\text{im}(f))$ is a Hilbertian module.

1.4. Let $\langle \cdot, \cdot \rangle$ be an admissible scalar product on M . Then $\langle \cdot, \cdot \rangle$ must be compatible with the topology on M and with the \mathcal{A} -action. The last condition means that the involution on \mathcal{A} determined by the scalar product $\langle \cdot, \cdot \rangle$ coincides with the involution of the von Neumann algebra \mathcal{A} :

$$\langle \lambda \cdot v, w \rangle = \langle v, \lambda^* \cdot w \rangle \quad (1)$$

for any $v, w \in M$, $\lambda \in \mathcal{A}$.

Suppose now that $\langle \cdot, \cdot \rangle_1$ is another admissible scalar product. Then there exists an operator

$$A : M \rightarrow M$$

such that

$$\langle v, w \rangle_1 = \langle Av, w \rangle \quad (2)$$

for any $v, w \in M$. The operator A has to be:

- (α) a linear homeomorphism (since the scalar products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_1$ define the same topology);
- (β) self-adjoint;
- (γ) positive;
- (δ) commuting with the \mathcal{A} -action.

In order to prove the last property, write

$$\begin{aligned}\langle \lambda \cdot v, w \rangle_1 &= \langle v, \lambda^* \cdot w \rangle_1, \\ \langle A(\lambda \cdot v), w \rangle &= \langle A(v), \lambda^* \cdot w \rangle, \\ \langle A(\lambda \cdot v), w \rangle &= \langle \lambda \cdot A(v), w \rangle\end{aligned}$$

for any $v, w \in M$, $\lambda \in \mathcal{A}$. Thus, it follows that $A(\lambda v) = \lambda A(v)$.

We conclude: *once an admissible scalar product on a Hilbertian module M has been chosen, there is a one-to-one correspondence between the admissible scalar products on M and the operators $A : M \rightarrow M$ satisfying $(\alpha) - (\delta)$ above.*

1.5. Corollary. *If $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_1$ are two admissible scalar products on a Hilbertian module M then the Hilbert modules $(M, \langle \cdot, \cdot \rangle)$ and $(M, \langle \cdot, \cdot \rangle_1)$ are isomorphic.*

Proof. Let $A : M \rightarrow M$ be the operator such that $\langle v, w \rangle_1 = \langle Av, w \rangle$ for $v, w \in M$. Let $B : M \rightarrow M$ be the positive square root of A ; since the spectrum of A is real and positive, the functional calculus produces the operator B uniquely. The operator B is positive, invertible, and commutes with the action of \mathcal{A} . Hence the map $x \mapsto Bx$, $x \in M$ establishes an isomorphism $(M, \langle \cdot, \cdot \rangle_1) \rightarrow (M, \langle \cdot, \cdot \rangle)$. \square

1.6. Using this corollary we may define *finitely generated Hilbertian modules* as those for which the corresponding Hilbert modules (obtained by a choice of an admissible scalar product) are finitely generated.

Note that the *von Neumann dimension* (denoted $\dim_\tau(M)$ or $\tau(M)$) of a Hilbertian module is also correctly defined (by virtue of the Corollary 1.5 above).

1.7. The commutant. Let M be a Hilbertian \mathcal{A} -module. Let $\mathcal{B} = \mathcal{B}(M)$ denote the algebra $\mathcal{B} = B_{\mathcal{A}}(M)$ of all bounded linear operators on M commuting with \mathcal{A} (the *commutant* of M).

Any choice of an admissible scalar product $\langle \cdot, \cdot \rangle$ on M , defines obviously a $*$ -operator on \mathcal{B} (by assigning to an operator its adjoint); this turns \mathcal{B} into a von Neumann algebra. Note that this involution $*$ depends on the scalar product $\langle \cdot, \cdot \rangle$ on M ; if we choose another admissible scalar product $\langle \cdot, \cdot \rangle_1$ on M , then the new involution will be given by

$$f \mapsto A^{-1} f^* A \quad \text{for } f \in \mathcal{B}, \quad (3)$$

where $A \in \mathcal{B}$ is the operator defined by $\langle v, w \rangle_1 = \langle Av, w \rangle$ for $v, w \in M$.

Our aim now is to construct canonically a trace on \mathcal{B} , using the given trace τ on \mathcal{A} .

1.8. Proposition. *If M is a finitely generated Hilbertian module, then the trace $\tau : \mathcal{A} \rightarrow \mathbb{C}$ determines canonically a trace on the commutant*

$$\text{Tr}_\tau : \mathcal{B} = \mathcal{B}(M) \rightarrow \mathbb{C} \quad (4)$$

which is finite, normal, and faithful (with respect to the involution on \mathcal{B} corresponding to any choice of an admissible scalar product on M). If M and N are two finitely

generated modules over \mathcal{A} , then the canonical traces Tr_τ on $\mathcal{B}(M)$, $\mathcal{B}(N)$ and on $\mathcal{B}(M \oplus N)$ are compatible in the following sense:

$$\text{Tr}_\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Tr}_\tau(A) + \text{Tr}_\tau(D), \quad (5)$$

for all $A \in \mathcal{B}(M)$, $D \in \mathcal{B}(N)$ and any morphisms $B : M \rightarrow N$, and $C : N \rightarrow M$.

Proof. The proof is straightforward and therefore we will indicate only the main steps.

Suppose first that M is free, that is, M is isomorphic to $l^2(\mathcal{A}) \otimes \mathbb{C}^k$ for some k . Then the commutant $\mathcal{B}(M)$ can be identified with the algebra of $k \times k$ -matrices with entries in \mathcal{A} , acting from the right on $l^2(\mathcal{A}) \otimes \mathbb{C}^k$ (the last module is viewed as the set of row-vectors with entries in $l^2(\mathcal{A})$). If $\alpha \in \mathcal{B}$ is represented by a $k \times k$ matrix (α_{ij}) , then one defines

$$\text{Tr}_\tau(\alpha) = \sum_{i=1}^k \tau(\alpha_{ii}).$$

This gives a trace on \mathcal{B} which satisfies all necessary conditions.

If M is not free, then we can embed it in a free module as a closed \mathcal{A} -invariant subspace. Then the commutant $\mathcal{B}(M)$ can be identified with a left ideal in the $k \times k$ -matrix algebra with entries in \mathcal{A} and the trace described in the previous paragraph restricts to this ideal and determines a trace on $\mathcal{B}(M)$. One then shows that the obtained trace on $\mathcal{B}(M)$ does not depend on the embedding of M in a free module. \square

1.9. Let M be a finitely generated Hilbertian module over a von Neumann algebra \mathcal{A} , as above. Denote by $\text{GL}(M)$ the group of all invertible elements of the commutant $\mathcal{B}(M) = B_{\mathcal{A}}(M)$. We will consider the norm topology on $\text{GL}(M)$; with this topology it is a Banach Lie group. Its Lie algebra can be identified with the commutant $\mathcal{B}(M)$. The canonical trace Tr_τ on the commutant $\mathcal{B}(M)$ (described in the previous subsection) is a homomorphism of the Lie algebra $\mathcal{B}(M)$ into the abelian Lie algebra \mathbb{C} . By the standard theorems, it defines a group homomorphism of the universal covering group of $\text{GL}(M)$ into \mathbb{C} . This approach leads to following construction of the Fuglede-Kadison determinants, compare [HS].

1.10. Theorem. *There exists a function*

$$\text{Det}_\tau : \text{GL}(M) \rightarrow \mathbb{R}^{>0} \quad (7)$$

(called the Fuglede - Kadison determinant), satisfying:

(a) Det_τ is a group homomorphism, that is,

$$\text{Det}_\tau(AB) = \text{Det}_\tau(A) \cdot \text{Det}_\tau(B) \quad (8)$$

for $A, B \in \text{GL}(M)$;

(b)

$$\text{Det}_\tau(\lambda I) = |\lambda|^{\dim_{\mathcal{A}}(M)} \quad (9)$$

for $\lambda \in \mathbb{C}$, $\lambda \neq 0$; here $I \in \text{GL}(M)$ denotes the identity operator;

(c)

$$\text{Det}_{\lambda\tau}(A) = \text{Det}_{\tau}(A)^{\lambda} \quad (10)$$

for $\lambda \in \mathbb{R}^{>0}$;

(d) Det_{τ} is continuous as a map $\text{GL}(M) \rightarrow \mathbb{R}^{>0}$, where $\text{GL}(M)$ is supplied with the norm topology;

(e) If A_t for $t \in [0, 1]$ is a continuous piecewise smooth path in $\text{GL}(M)$ then

$$\log\left[\frac{\text{Det}_{\tau}(A_1)}{\text{Det}_{\tau}(A_0)}\right] = \int_0^1 \Re \text{Tr}_{\tau}[A_t^{-1} A'_t] dt. \quad (11)$$

Here \Re denotes the real part, Tr_{τ} denotes the canonical trace on the commutant constructed in Proposition 1.8 and A'_t denotes the derivative of A_t with respect to t .

(f) Let M and N be two finitely generated modules over \mathcal{A} , and $A \in \text{GL}(M)$ and $B \in \text{GL}(N)$ two invertible automorphisms, and let $\gamma : N \rightarrow M$ be a homomorphism. Then the map given by the matrix

$$\begin{pmatrix} A & \gamma \\ 0 & B \end{pmatrix}$$

belongs to $\text{GL}(M \oplus N)$ and

$$\text{Det}_{\tau} \begin{pmatrix} A & \gamma \\ 0 & B \end{pmatrix} = \text{Det}_{\tau}(A) \cdot \text{Det}_{\tau}(B) \quad (12)$$

Proof. We are going to accept the following definition. Given an invertible operator $A \in \text{GL}(M)$, find a continuous piecewise smooth path $A_t \in \text{GL}(M)$ with $t \in [0, 1]$, such that $A_0 = I$ and $A_1 = A$ (it is well known that the group $\text{GL}(M)$ is pathwise connected, cf. [Di]). Then define

$$\log \text{Det}_{\tau}(A) = \int_0^1 \Re \text{Tr}_{\tau}[A_t^{-1} A'_t] dt. \quad (13)$$

We want to show that the integral does not depend on the choice of the path, joining A with the identity I . Consider the integrals of the form

$$\int_a^b \text{Tr}_{\tau}[A_t^{-1} A'_t] dt, \quad (14)$$

where A_t for $t \in [a, b]$ is a piecewise smooth path in $\text{GL}(M)$. Suppose first that the path A_t is "small" in the following sense: for all $t \in [a, b]$ we have $\|A_a^{-1} A_t - I\| < 1$. Then one may find a piecewise smooth path $C_t \in \mathcal{B}(M)$ such that $A_t = A_a \exp(C_t)$ and $C_a = 0$; here C_t is defined as $\log(A_a^{-1} A_t)$ and the logarithm function defined by its Taylor power expansion around 1. Using Duhamel's formula

$$A'_t = A_a \int_0^1 e^{(1-s)C_t} C'_t e^{sC_t} ds \quad (15)$$

one computes

$$\begin{aligned}
\int_a^b \mathrm{Tr}_\tau[A_t^{-1}A'_t]dt &= \int_a^b \mathrm{Tr}_\tau(C'_t)dt = \\
&= \int_a^b \frac{d}{dt}(\mathrm{Tr}_\tau(C_t))dt = \mathrm{Tr}_\tau(C_b) = \\
&= \mathrm{Tr}_\tau(\log(A_a^{-1}A_b))
\end{aligned} \tag{16}$$

If A_t is an arbitrary piecewise smooth path in $\mathrm{GL}(M)$, defined for $t \in [0, 1]$, (which is now not supposed to be "small"), then the interval $[0, 1]$ can be divided into subintervals $t_0 = 0 < t_1 < \dots < t_N = 1$ such that $\|A_{t_i}^{-1}A_t - I\| < 1$ for all $t \in [t_i, t_{i+1}]$. Thus by the argument above we have

$$\int_a^b \mathrm{Tr}_\tau[A_t^{-1}A'_t]dt = \sum_{i=0}^{N-1} \mathrm{Tr}_\tau(\log(A_{t_i}^{-1}A_{t_{i+1}})) \tag{17}$$

This shows, in particular, that the integral (14) depends only on the homotopy class of the path (relative to the end points). One easily checks the following homomorphism property: if A_t and B_t are two piecewise smooth paths in $\mathrm{GL}(M)$ defined for $t \in [a, b]$ and if $C_t = A_t B_t$ is their product then

$$\int_a^b \mathrm{Tr}_\tau[C_t^{-1}C'_t]dt = \int_a^b \mathrm{Tr}_\tau[A_t^{-1}A'_t]dt + \int_a^b \mathrm{Tr}_\tau[B_t^{-1}B'_t]dt. \tag{18}$$

Now we observe that the integral

$$\int_a^b \Re \mathrm{Tr}_\tau[A_t^{-1}A'_t]dt, \tag{19}$$

taken along any closed loop in $\mathrm{GL}(M)$, vanishes. This follows from the fact that any closed loop in $\mathrm{GL}(M)$ is homotopic to a product of loops of the form

$$A_t = \exp(2\pi i t S), \quad \text{for } 0 \leq t \leq 1, \tag{20}$$

where $S \in \mathcal{B}(M)$ and $\mathrm{Tr}_\tau(S)$ is real; for loops of this form the integral vanishes obviously. In fact, if we choose an admissible scalar product on M then the commutant $\mathcal{B}(M)$ becomes a finite von Neumann algebra; the result of H. Araki, M-S.B. Smith and L. Smith (cf. [ASS], Theorem 2.8) states that any loop in the group $\mathcal{U}(M)$ of unitary elements of $\mathcal{B}(M)$ is homotopic to a product of the loops of the form (20) where S is a self-adjoint element of $\mathcal{B}(M)$. Using the polar decomposition it is easy to construct a deformation retraction of $\mathrm{GL}(M)$ onto $\mathcal{U}(M)$ and thus the result follows.

This gives a construction of the FK determinant Det_τ . The homomorphism property (18) proves multiplicativity of the determinants (a). The other properties (b), (c), (d), (e) are clearly satisfied. We are left to prove (f). Let A_t and B_t be piecewise

smooth paths connecting A and B with the identity maps of M and N correspondingly. Then the family of maps $\begin{pmatrix} A_t & t\gamma \\ 0 & B_t \end{pmatrix}$ joins the given map with the identity of $M \oplus N$ and we obtain

$$\begin{aligned} \mathrm{Tr}_\tau \left(\begin{pmatrix} A_t & t\gamma \\ 0 & B_t \end{pmatrix}^{-1} \cdot \begin{pmatrix} A'_t & \gamma \\ 0 & B'_t \end{pmatrix} \right) &= \\ \mathrm{Tr}_\tau \left(\begin{pmatrix} A_t^{-1} & -B_t^{-1}t\gamma A_t^{-1} \\ 0 & B_t^{-1} \end{pmatrix} \cdot \begin{pmatrix} A'_t & \gamma \\ 0 & B'_t \end{pmatrix} \right) &= \\ \mathrm{Tr}_\tau \begin{pmatrix} A'_t A_t^{-1} & * \\ 0 & B'_t B_t^{-1} \end{pmatrix} &= \\ \mathrm{Tr}_\tau(A'_t A_t^{-1}) + \mathrm{Tr}_\tau(B'_t B_t^{-1}) \end{aligned}$$

where the formula (5) of Proposition 1.8 has been used. \square

1.11. As an example of computation of the Fuglede-Kadison determinant based on the above definition, consider the following situation. Suppose that an operator $A \in \mathrm{GL}(M)$ is given by

$$A = \int_0^\infty \lambda dE_\lambda \quad (21)$$

where λ is real and E_λ is a spectral projection in $\mathcal{B}(M)$. We will assume that the operator A is invertible in $\mathcal{B}(M)$; this means that $E_\lambda = 0$ for sufficiently small λ . Then we can choose the straight path

$$A_t = t(A - I) + I, \quad t \in [0, 1]$$

as the path joining A with I inside $\mathrm{GL}(M)$. Applying the above definition we obtain

$$\begin{aligned} \log \mathrm{Det}_\tau(A) &= \int_0^1 \Re \mathrm{Tr}_\tau[A_t^{-1} A'_t] dt = \\ &= \int_0^\infty \left[\int_0^1 \frac{\lambda - 1}{t(\lambda - 1) + 1} dt \right] d\phi_\lambda = \int_0^\infty \log \lambda d\phi_\lambda, \end{aligned} \quad (22)$$

where $\phi_\lambda = \mathrm{Tr}_\tau E_\lambda$ is the spectral density function.

§2. Determinant line of a Hilbertian module

In this section we describe a construction of the determinant line associated to a finitely generated Hilbertian module. It will be used later in §3 in the constructions of combinatorial and analytic torsion invariants associated to polyhedra and compact manifolds.

Let \mathcal{A} be a finite von Neumann algebra with a fixed finite, normal, and faithful trace $\tau : \mathcal{A} \rightarrow \mathbb{C}$. Given a Hilbertian module M of finite type over \mathcal{A} , we are going to associate with a M (in a canonical way) an oriented real line which we will denote $\det(M)$; we will call it the *determinant line of M* . Our construction will generalize the determinant line of a finite dimensional vector space.

In order to clarify the precise definition (given in the paragraph below) let's make the following remark. In the category of finite dimensional vector spaces, the determinant line consists of volume forms. What is the correct generalization of the notion of volume form for Hilbertian modules? Note that one can define a volume form on a vector space by presenting a scalar product on this vector space and two different scalar products determine the same volume form if and only if the determinant of the transition operator (determined by the pair of scalar products) is equal to 1. Thus, we may consider a volume form as an equivalence class of admissible scalar products. In fact, in the last form, the notion of volume form can be generalized to Hilbertian modules over von Neumann algebras, using the FK determinants.

2.1. Define $\det(M)$ as a real vector space generated by symbols $\langle \cdot, \cdot \rangle$, one for any admissible scalar product on M , subject to the following relations: for any pair $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ of admissible scalar products on M we write the following relation

$$\langle \cdot, \cdot \rangle_2 = \text{Det}_\tau(A)^{-1/2} \cdot \langle \cdot, \cdot \rangle_1, \quad (23)$$

where $A \in \text{GL}(M)$ is such that

$$\langle v, w \rangle_2 = \langle Av, w \rangle_1$$

for all $v, w \in M$. Here the transition operator A is invertible and belongs to the commutant $\mathcal{B}(M)$ (cf. section 1) and $\text{Det}_\tau(A)$ denotes the Fuglede-Kadison determinant of A constructed (in subsection 1.10) with the aid of the canonical trace Tr_τ on the commutant.

Assume that we have three different admissible scalar products $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ and $\langle \cdot, \cdot \rangle_3$ on M . Suppose that

$$\langle v, w \rangle_2 = \langle Av, w \rangle_1 \quad \text{and} \quad \langle v, w \rangle_3 = \langle Bv, w \rangle_2$$

for all $v, w \in \mathcal{B}(M)$, where $A, B \in \mathcal{B}(M)$, then $\langle v, w \rangle_3 = \langle ABv, w \rangle_1$ and in $\det(M)$ we have the relations

$$\langle \cdot, \cdot \rangle_2 = \text{Det}_\tau(A)^{-1/2} \cdot \langle \cdot, \cdot \rangle_1,$$

$$\langle \cdot, \cdot \rangle_3 = \text{Det}_\tau(B)^{-1/2} \cdot \langle \cdot, \cdot \rangle_2,$$

$$\langle \cdot, \cdot \rangle_3 = \text{Det}_\tau(AB)^{-1/2} \cdot \langle \cdot, \cdot \rangle_1,$$

and the third relation follows from the first two via the homomorphism property of the Fuglede-Kadison determinant.

Thus we obtain, that $\det(M)$ is one-dimensional real vector space generated by the symbol $\langle \cdot, \cdot \rangle$ of any admissible scalar product on M .

Note also, that the real line $\det(M)$ has the canonical orientation, since the transition coefficients $\text{Det}_\tau(A)^{-1/2}$ are always positive. Thus we may speak of *positive* and *negative* elements of $\det(M)$. The set of all positive elements of $\det(M)$ will be denoted $\det_+(M)$.

We will think of elements of $\det(M)$ as "volume forms" on M .

If M is trivial module, $M = 0$, then we set $\det(M) = \mathbb{R}$, by definition.

To illustrate our definitions, consider the case $\mathcal{A} = \mathbb{C}$ with the standard trace. Now M is just a finite dimensional vector space over \mathbb{C} . Any scalar product $\langle \cdot, \cdot \rangle$ on M determines an element

$$e_1 \wedge e_2 \wedge \cdots \wedge e_n \in \Lambda_{\mathbb{C}}^n(M).$$

Here $n = \dim_{\mathbb{C}} M$, $\Lambda_{\mathbb{C}}^n(M)$ denotes the highest exterior power of M and e_1, e_2, \dots, e_n is an orthonormal basis of M with respect to $\langle \cdot, \cdot \rangle$. We obtain a map $\det_+(M) \rightarrow |\Lambda_{\mathbb{C}}^n(M)| = \Lambda^n(M)/\sim$; the equivalence relation \sim is $v \sim w$ iff $v = e^{i\phi}w$ for some $\phi \in \mathbb{R}$. This map is well defined; to show this one observes that in finite dimensional case the Fuglede - Kadison determinant coincides with the absolute value of the usual determinant. Also, if we multiply the scalar product $\langle \cdot, \cdot \rangle$ by λ^2 , where $\lambda > 0$, then the new orthonormal basis will be $\lambda^{-1}e_1, \lambda^{-1}e_2, \dots, \lambda^{-1}e_n$ and the corresponding element of $\Lambda_{\mathbb{C}}^n(M)$ will be $\lambda^{-n}e_1 \wedge e_2 \wedge \cdots \wedge e_n$; this is compatible with (23) and shows that the map $\det_+(M) \rightarrow |\Lambda_{\mathbb{C}}^n(M)|$ is \mathbb{R}_+ -linear. Thus, we obtain that in finite dimensions we have an identification $\det_+(M) \simeq |\Lambda_{\mathbb{C}}^n(M)|$.

2.2. Given two finitely generated Hilbertian modules M and N over \mathcal{A} , and a pair $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_N$ of admissible scalar products on M and N correspondingly, we may obviously define the scalar product $\langle \cdot, \cdot \rangle_M \oplus \langle \cdot, \cdot \rangle_N$ on the direct sum $M \oplus N$. This defines an isomorphism

$$\det(M) \otimes \det(N) \rightarrow \det(M \oplus N). \quad (24)$$

Using Theorem 1.10, it is easy to show that *this homomorphism is canonical*, that is, it does not depend on the choice of the metrics $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_N$. From the description given above it is clear that the homomorphism (22) preserves the orientations.

2.3. Note that, *any isomorphism $f : M \rightarrow N$ between finitely generated Hilbertian modules induces canonically an isomorphism of the determinant lines*

$$f_* : \det(M) \rightarrow \det(N). \quad (25)$$

Moreover, the induced map f_ preserves the orientations of the determinant lines.* Indeed, if $\langle \cdot, \cdot \rangle_M$ is an admissible scalar product on M , then we set

$$f_*(\langle \cdot, \cdot \rangle_M) = \langle \cdot, \cdot \rangle_N, \quad (26)$$

where $\langle \cdot, \cdot \rangle_N$ is the scalar product on N given by $\langle v, w \rangle_N = \langle f^{-1}(v), f^{-1}(w) \rangle_M$ for $v, w \in N$ (this scalar product is admissible since f is an isomorphism). This definition does not depend on the choice of the scalar product $\langle \cdot, \cdot \rangle_M$ on M : if we have a different admissible scalar product $\langle \cdot, \cdot \rangle'_M$ on M , where $\langle v, w \rangle'_M = \langle A(v), w \rangle_M$ with $A \in \text{GL}(M)$, then the induced scalar product on N will be

$$\langle v, w \rangle'_N = \langle (f^{-1}Af)v, w \rangle_N$$

and our statement follows from property (a) of the Fuglede-Kadison determinant, cf. Theorem 1.10.

2.4. It is obvious from the definition, that our construction is *functorial*: if $f : M \rightarrow N$ and $g : N \rightarrow L$ are two isomorphisms between finitely generated Hilbertian modules then

$$(g \circ f)_* = g_* \circ f_* \quad (27)$$

2.5. Proposition. *If $f : M \rightarrow M$ is an automorphism of a finitely generated Hilbertian module M , $f \in \text{GL}(M)$, then the induced homomorphism $f_* : \det(M) \rightarrow \det(M)$ coincides with the multiplication by $\text{Det}_\tau(f) \in \mathbb{R}^{>0}$.*

Proof. Let $\langle \cdot, \cdot \rangle$ be an admissible scalar product on M , then the induced by f scalar product $\langle \cdot, \cdot \rangle'$ is given by

$$\langle v, w \rangle' = \langle f^{-1}(v), f^{-1}(w) \rangle = \langle (ff^*)^{-1}v, w \rangle.$$

Thus in $\det(M)$ we have

$$\langle \cdot, \cdot \rangle' = \sqrt{\text{Det}_\tau(ff^*)} \cdot \langle \cdot, \cdot \rangle = \text{Det}_\tau(f) \cdot \langle \cdot, \cdot \rangle.$$

2.6. Proposition. *Any exact sequence*

$$0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0 \quad (28)$$

of finitely generated Hilbertian modules determines canonically an isomorphism

$$\det(M') \otimes \det(M'') \rightarrow \det(M), \quad (29)$$

preserving the orientations.

Proof. As is well known, any such exact sequence splits. Choose a splitting

$$0 \rightarrow M' \xleftarrow{r} M \xleftarrow{s} M'' \rightarrow 0.$$

Then any pair of admissible scalar products $\langle \cdot, \cdot \rangle_{M'}$ and $\langle \cdot, \cdot \rangle_{M''}$ on M' and M'' correspondingly, define the following admissible scalar product on M :

$$\langle v, w \rangle_M = \langle r(v), r(w) \rangle_{M'} + \langle \beta(v), \beta(w) \rangle_{M''} \quad (30)$$

This defines an isomorphism

$$\det(M') \otimes \det(M'') \rightarrow \det(M).$$

This isomorphism can be also described as the composition

$$\det(M') \otimes \det(M'') \rightarrow \det(M' \oplus M'') \xrightarrow{(\alpha \oplus s)_*} \det(M)$$

where the first map is given by (24). To finish the proof we only have to show independence on the splitting s . If s' is another splitting then it can be represented in the form

$$s' = s + \alpha \circ \gamma, \quad (31)$$

where $\gamma : M'' \rightarrow M'$ is a homomorphism. Now the result follows from the commutative diagram

$$\begin{array}{ccc} M' \oplus M'' & \xrightarrow{\alpha \oplus s} & M \\ \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix} \right) \uparrow & & \uparrow \\ M' \oplus M'' & \xrightarrow{\alpha \oplus s'} & M \end{array} \quad (32)$$

and from the fact that $\text{Det}_\tau \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix} \right) = 1$, cf. Theorem 1.10, statement (f). \square

§3. D -admissible scalar products, D -isomorphisms, and D -exact sequences

In this section, we consider generalisations of the notions and results of section 2. We introduce the notion of a D -admissible scalar product on a finitely generated Hilbertian module (here the letter D refers to the "*determinant class*" type condition, following the terminological convention suggested in [BFKM]). We show that any D -admissible scalar product defines a nonzero element in the determinant line. We also introduce the notion of D -isomorphism and D -exact sequences. We prove that any D -isomorphism induces an isomorphism of the determinant lines. Under the hypothesis that the chain complex is of determinant class, we construct a natural isomorphism between the determinant line of the chain complex and the determinant line of its reduced L^2 homology.

3.1. Consider a Hilbertian module M of finite type over \mathcal{A} (as above). A scalar product $\langle \cdot, \cdot \rangle$ on M will be called *D -admissible* if it can be represented in the form

$$\langle v, w \rangle = \langle A(v), w \rangle_1 \quad \text{for } v, w \in M,$$

where $\langle \cdot, \cdot \rangle_1$ is an admissible scalar product on M and $A \in \mathcal{B}_{\mathcal{A}}(M)$ is an injective (possibly not invertible) homomorphism $A : M \rightarrow M$, which is positive and self-adjoint with respect to $\langle \cdot, \cdot \rangle_1$, and the following property is satisfied: if

$$A = \int_0^\infty \lambda dE_\lambda \tag{33}$$

is the spectral decomposition of A and if

$$\phi(\lambda) = \dim_\tau(E_\lambda) = \text{Tr}_\tau(E_\lambda) \tag{34}$$

denotes the corresponding spectral density function, then the integral

$$\int_0^\infty \ln(\lambda) d\phi(\lambda) > -\infty \tag{35}$$

is assumed to converge to a finite value. Note that the integral (35) may only diverge at point $\lambda = 0$.

We want to show that *any D -admissible scalar product determines canonically a nonzero element of the determinant line $\det(M)$.*

3.2. Proposition. *In the above notations, the non-zero element*

$$\exp[-1/2 \int_0^\infty \ln(\lambda) d\phi(\lambda)] \cdot \langle \cdot, \cdot \rangle_1 \tag{36}$$

of the determinant line $\det(M)$ depends only on the D -admissible scalar product $\langle \cdot, \cdot \rangle$ and does not depend on the choice of the scalar product $\langle \cdot, \cdot \rangle_1$. The class of this element in $\det(M)$ will be denoted by the same symbol $\langle \cdot, \cdot \rangle$.

Introduce the following notation

$$\text{Det}_\tau(A) = \exp\left[\int_0^\infty \ln(\lambda) d\phi(\lambda)\right]; \tag{37}$$

it is an extension of the Fuglede-Kadison determinant to some non-invertible self-adjoint positive operators, which was discussed in [FK],[Lu] (compare (22)). With this notation the formula above can be written as

$$\langle \cdot, \cdot \rangle = \text{Det}_\tau(A)^{-1/2} \cdot \langle \cdot, \cdot \rangle_1, \quad (38)$$

which is consistent with our previous constructions.

Proof of Proposition 3.2. Suppose that $\langle \cdot, \cdot \rangle_2$ is another admissible scalar product on M and let $\langle v, w \rangle = \langle Bv, w \rangle_2$ for $v, w \in M$, where $B \in \mathcal{B}(M)$. Then there is an invertible $C \in \text{GL}(M)$, such that $\langle v, w \rangle_2 = \langle Cv, w \rangle_1$, where $C \in \text{GL}(M)$. Hence $A = CB$ and we obtain

$$\text{Det}_\tau(B)^{-1/2} \cdot \langle \cdot, \cdot \rangle_2 = \text{Det}_\tau(B)^{-1/2} \cdot \text{Det}_\tau(C)^{-1/2} \cdot \langle \cdot, \cdot \rangle_1 = \text{Det}_\tau(A)^{-1/2} \cdot \langle \cdot, \cdot \rangle_1$$

since $\text{Det}_\tau(A) = \text{Det}_\tau(C) \text{Det}_\tau(B)$, cf. [FK], [Lu]. \square

3.3. Definition. An injective with dense image homomorphism $f : M \rightarrow N$ of Hilbertian modules will be called *D-isomorphism* if for some (and hence for any) admissible scalar product $\langle \cdot, \cdot \rangle_N$ on N the induced scalar product $\langle \cdot, \cdot \rangle_M$ on M (which is given by $\langle v, w \rangle_M = \langle f(v), f(w) \rangle_N$) is *D-admissible*.

Any *D-isomorphism* $f : M \rightarrow N$ induces an isomorphism

$$f_* : \det(M) \rightarrow \det(N),$$

where $f^*(\langle \cdot, \cdot \rangle_M) = \langle \cdot, \cdot \rangle_N$ in the above notations.

3.4. Definition. A sequence of Hilbertian modules and homomorphisms

$$0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0 \quad (39)$$

will be called *D-exact* if α is a monomorphism, $\text{im}(\alpha) = \ker(\beta)$, and the induced by β map $M/\ker(\beta) \rightarrow M''$ is a *D-isomorphism* in the sense of Definition 3.3.

We can slightly generalize Proposition 2.6:

3.5. Proposition. *Any D-exact sequence*

$$0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$$

determines canonically an isomorphism

$$\det(M') \otimes \det(M'') \rightarrow \det(M). \quad (40)$$

Proof. Denote $\bar{M} = M/\ker(\beta)$ and let $\bar{\beta} : \bar{M} \rightarrow M''$ be the homomorphism determined by β . It is a *D-isomorphism* and so by the remark above it induced an isomorphism

$$\bar{\beta}_* : \det(\bar{M}) \rightarrow \det(M'').$$

Thus we obtain the following isomorphism

$$\det(M') \otimes \det(M'') \xrightarrow{\text{id} \otimes \bar{\beta}_*^{-1}} \det(M') \otimes \det(\bar{M}) \rightarrow \det(M),$$

where the last isomorphism is given by proposition 2.6, applied to the exact sequence

$$0 \rightarrow M' \xrightarrow{\alpha} M \rightarrow \bar{M} \rightarrow 0.$$

This completes the proof.

3.6. Let $M_* = \oplus M_i$ be a *graded Hilbertian \mathcal{A} -module of finite type*. This means that each M_i is finitely generated and there are only finitely many nonzero modules M_i . We define the determinant line of M in the usual way:

$$\det(M) = \otimes \det(M_i)^{(-1)^i}, \quad (41)$$

where $\det(M_i)^{-1}$ denotes the dual line of $\det(M_i)$.

3.7. A crucial role in the finite dimensional linear algebra of determinant lines is played by the canonical isomorphism between the determinant line of a chain complex and the determinant line of its homology, cf. [BGS]. The similar statement in the category of Hilbertian modules *is not true in general*, but it is true under an additional requirement, which we are going to describe now. Roughly speaking, this condition means that *the reduced L^2 cohomology "properly represents" the "full" cohomology*. Recall that the reduced L^2 cohomology is the factor of the submodule of cycles by the *closure* of the submodule of the boundaries, cf. (45).

3.8. Definition. Let

$$0 \rightarrow C_N \xrightarrow{\partial} C_{N-1} \xrightarrow{\partial} \dots C_0 \rightarrow 0 \quad (42)$$

be a chain complex of finite length formed by finitely generated Hilbertian modules and bounded linear maps ∂ commuting with the \mathcal{A} -action. Let Z_i denote the submodule of cycles and let B_i denote the submodule of boundaries. Following [BFKM], we will say that the chain complex (42) belongs to *the determinant class* if the following sequence

$$0 \rightarrow Z_i \rightarrow C_i \rightarrow \bar{B}_{i-1} \rightarrow 0 \quad (43)$$

is *D-exact*, cf. 3.4 above.

Note, that the above condition *imposes no restrictions on the size of the reduced L^2 -homology Z_i/\bar{B}_i* , which may be arbitrary large. This condition can be expressed through the density functions of the Laplacians of different dimensions; it is satisfied if the Novikov-Shubin invariants are all positive.

More precisely, the property of chain complex (42) to be of determinant class *depends only on the torsion part of its extended L^2 homology*, as defined in [F1], [F2]. In fact, the torsion part of the extended L^2 homology of complex (42) in dimension $i-1$ equals to $(\partial : C_i \rightarrow \bar{B}_{i-1})$, cf. formulae (16) and (18) in [F2], and our statement follows by comparing the Definitions 3.1, 3.3, 3.4, and 3.8. The proof of the fact that the property (35) depends only on the *isomorphism type* of the torsion part of the extended homology (viewed as an object of the extended abelian category) goes as follows. First, one refers to theorem of Gromov and Shubin [GS] to obtain that the dilatational equivalence class of the spectral density function is an isomorphism type invariant of a torsion object (cf. also [F2], Proposition 4.5). Secondly, one may use Lemma 1.20 of [BFKM], for example, to arrive at the desired conclusion.

Since the extended L^2 homology (up to isomorphism) depends only on the homotopy type of the chain complex, we obtain (cf. also [BFKM], Proposition 5.6):

3.9. Corollary. *The property of a chain complex C of finitely generated Hilbertian modules to be of the determinant class depends only on the homotopy type of C in the category of finitely generated complexes of Hilbertian modules over \mathcal{A} .*

Now we will formulate the generalization mentioned above.

3.10. Proposition. *In the category of finite chain complexes*

$$C : (0 \rightarrow C_N \xrightarrow{\partial} C_{N-1} \xrightarrow{\partial} \dots C_0 \rightarrow 0) \quad (43)$$

of finitely generated Hilbertian modules of determinant class there is a natural isomorphism between the determinant lines of graded Hilbertian modules

$$\phi_C : \det(C) \rightarrow \det(H_*), \quad (44)$$

where $H_ = \oplus H_i$ is the graded Hilbertian module consisting of the reduced L^2 -homology of C , that is,*

$$H_i = \ker(\partial) / \text{cl}(\text{im}(\partial)). \quad (45)$$

Proof. Consider the following two sequences:

$$0 \rightarrow \bar{B}_i \rightarrow Z_i \rightarrow H_i \rightarrow 0$$

and

$$0 \rightarrow Z_i \rightarrow C_i \rightarrow \bar{B}_{i-1} \rightarrow 0.$$

The first sequence is exact and the second sequence is D -exact. By propositions 2.6 and 3.5 above we have natural isomorphisms

$$\det(Z_i) \rightarrow \det(H_i) \otimes \det(\bar{B}_i)$$

and

$$\det(C_i) \rightarrow \det(Z_i) \otimes \det(\bar{B}_{i-1}).$$

Together they give the natural isomorphism

$$\det(C_i) \rightarrow \det(H_i) \otimes \det(\bar{B}_i) \otimes \det(\bar{B}_{i-1})$$

and thus we obtain the natural isomorphism

$$\phi_C : \det(C) = \otimes_i \det(C_i)^{(-1)^i} \rightarrow \otimes_i \det(H_i)^{(-1)^i} = \det(H_*).$$

This completes the proof. \square

Now we will compute numerically the canonical isomorphism (44).

3.11. Proposition. *Suppose that we have fixed an admissible scalar product on each chain space C_i of the chain complex (43) (which is assumed to be of determinant class). Let $\alpha_i \in \det(C_i)$ represent the induced volume form on C_i and let*

$$\alpha = \prod \alpha_i^{(-1)^i} \in \det(C_*)$$

be their alternating product. Let $\Delta_i = \partial^ \partial + \partial \partial^* : C_i \rightarrow C_i$ denote the Laplacian constructed by means of the chosen scalar products. The reduced homology $H_* = H_*(C)$ can now be identified with the space of harmonic forms via the Hodge decomposition.*

We will denote by $\beta \in \det(H_*)$ the volume form on the graded homology inherited by means of the above identification. Then the following formula holds

$$\phi_C = \prod_{i=0}^N \text{Det}_\tau(\Delta_i^+)^{(-1)^{i+1}/2} \cdot \alpha^* \otimes \beta \in \det(C)^{-1} \otimes \det(H_*), \quad (46)$$

where Δ_i^+ denotes the restriction of the Laplacian Δ_i on the orthogonal complement to the space of harmonic forms.

Proof. First we observe that the canonical isomorphism (44) behaves in a very simple way with respect to direct sums of chain complexes. Namely, $\phi_{C_1 \oplus C_2} = \phi_{C_1} \otimes \phi_{C_2}$. Now, any chain complex of Hilbertian modules can be represented as a direct sum of complexes of the form

$$(\cdots \rightarrow 0 \rightarrow C_i \xrightarrow{\partial} C_{i-1} \rightarrow 0 \rightarrow \cdots)$$

(only two chain spaces are non-zero and the boundary map is a D -isomorphism) and of chain complexes with zero differentials. Using the above remark one has to check (47) only for such complexes, which is straightforward. \square

Note, that the analogous formula for *cochain complexes* C (of determinant class) is slightly different

$$\phi_C = \prod_{i=0}^N \text{Det}_\tau(\Delta_i^+)^{(-1)^{i+1}/2} \cdot \alpha^* \otimes \beta \in \det(C)^{-1} \otimes \det(H^*), \quad (47)$$

cf. [BZ], [BFKM].

§4. Combinatorial L^2 -torsion

In this section we will define and study a generalization of the classical construction of the combinatorial torsion (of Reidemeister, Franz and DeRham) to the case of infinite dimensional representations, which are modules over a finite von Neumann algebra. Given a finite polyhedron K and an unimodular representation of its fundamental group in a module M over a finite von Neumann algebra \mathcal{A} , the torsion invariant defined here is a *positive element of the determinant line*

$$\det(M)^{-\chi(K)} \otimes \det(H_*(K, M)).$$

Under the assumption that the Euler characteristic of K vanishes (which is always the case if K is a closed manifold of odd dimension), the torsion does not depend on the choice of the volume form on M and it can be understood as a volume form on the reduced L^2 -homology $H_*(K, M)$, that is, as an element of the determinant line $\det(H_*(K, M))$.

4.1. Let K be a finite cell complex. Denote by $\pi = \pi_1(K)$ its fundamental group and by $C_*(\tilde{K})$ the cellular chain complex of the universal covering \tilde{K} of K . Note that the group π acts on $C_*(\tilde{K})$ from the left and $C_*(\tilde{K})$ is a finite complex of free $\mathbb{Z}[\pi]$ -modules with lifts of the cells of K representing a free basis of $C_*(\tilde{K})$ over $\mathbb{Z}[\pi]$.

Let \mathcal{A} be a fixed finite von Neumann algebra with a finite, normal and faithful trace τ . Let M be a finitely generated Hilbertian module over \mathcal{A} . As above, we will denote by $\mathcal{B}(M) = \mathcal{B}_{\mathcal{A}}(M)$ the commutant of M .

4.2. We will consider *representations of the group* π in M . Any such representation is a homomorphism $\pi \rightarrow \mathcal{B}(M) = \mathcal{B}_{\mathcal{A}}(M)$ of multiplicative groups. We will think of π as acting on M from the right (via the representation $\pi \rightarrow \mathcal{B}(M)$); thus M will have a structure of $(\mathcal{A} - \pi)$ -bimodule. In this situation we will say that M is *Hilbertian* $(\mathcal{A} - \pi)$ -bimodule.

We will say that a Hilbertian $(\mathcal{A} - \pi)$ -bimodule M is *unimodular* if for every element $g \in \pi$ the Fuglede-Kadison determinant

$$\text{Det}_{\tau}(g) = 1, \quad (48)$$

equals 1, where g is viewed as an invertible linear operator $M \rightarrow M$, given by the right multiplication by g .

We will say that a Hilbertian $(\mathcal{A} - \pi)$ -bimodule M is *unitary* if there exists an admissible scalar product $\langle \cdot, \cdot \rangle$ on M such that the action of π preserves this scalar product. Obviously, any unitary Hilbertian bimodule is unimodular.

Any Hilbertian $(\mathcal{A} - \pi)$ -bimodule M determines a *flat Hilbertian bundle over* K *with fiber* M . In fact, consider Borel's construction

$$\mathcal{E} = M \times_{\pi} \tilde{K}$$

together with the obvious projection map $\mathcal{E} \rightarrow K$; it has a canonical structure of a flat \mathcal{A} -Hilbertian bundle. Equivalently, it can be viewed as a locally free sheaf of Hilbertian \mathcal{A} -modules.

4.3. Examples. As a first concrete example, consider the classical case, when $\mathcal{A} = \mathcal{N}(\pi)$ is the von Neumann algebra of π and $M = \ell^2(\pi)$ is the completion of the group algebra of π with respect to the canonical trace on it, with \mathcal{A} acting on M from the left and with π acting on M from the right. This $(\mathcal{A} - \pi)$ -bimodule $M = \ell^2(\pi)$ is unitary and thus unimodular, as well.

As a more general example consider the following Hilbertian $(\mathcal{A} - \pi)$ -bimodule $M = \ell^2(\pi) \otimes_{\mathbb{C}} V$ where V is a finite dimensional unimodular representation of π . Here the left action of $\mathcal{A} = \mathcal{N}(\pi)$ on M is the same as the action on $\ell^2(\pi)$ and the right action of π is the diagonal action: $(x \otimes v)g = xg \otimes vg$ for $x \in \ell^2(\pi)$, $v \in V$, and $g \in \pi$.

4.4. Construction of the torsion. Given an unimodular $(\mathcal{A} - \pi)$ -bimodule M , we can form the complex

$$C_*(K, M) = M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K}). \quad (49)$$

It is a chain complex of finitely generated \mathcal{A} -modules; each chain group $C_i(K, M)$ can be identified with a finite direct sum of a number of copies of M and the number of summands equals the number of i -dimensional simplexes of K .

We will assume that *this chain complex* $C_*(K, M)$ *is of determinant class*, cf. 3.8 above. Then by Proposition 3.10 we obtain a natural isomorphism

$$\det(C_*(K, M)) \xrightarrow{\cong} \det(H_*(K, M)), \quad (50)$$

where $H_*(K, M)$ denotes *the reduced L^2 -homology* of $C_*(K, M)$. Now, under the unimodularity assumption there is natural isomorphism

$$\det(M)^{\chi(K)} \xrightarrow{\cong} \det(C_*(K, M)), \quad (51)$$

defined as follows. For any cell $e \subset K$ fix a lifting \tilde{e} of e in the universal covering. Then the cells \tilde{e} form a free $\mathbb{Z}\pi$ -basis of the complex $C_*(\tilde{K})$ and therefore they allow us to represent $C_*(K, M)$ as the direct sum of copies of M , one for each cell. Thus, the determinant line $\det(C_*(K, M))$ can be identified with $\det(M)^{\chi(K)}$, since the cells of odd dimension contribute negative factors of $\det(M)$ into the total determinant line.

We only have to show that this identification does not depend on the choice of the liftings \tilde{e} . Consider an arbitrary set of liftings of the cells of K . It can be obtained as follows. Suppose that for each cell $e \subset K$ an element $g_e \in \pi$ has been fixed. Then the cells $g_e \tilde{e}$ form another set of liftings. Since the Fuglede-Kadison determinant of the map

$$\oplus M \rightarrow \oplus M \quad (52)$$

(where in the sums on both sides the number of copies of M is equal to the number of the cells in K), given by the diagonal matrix with g_e on the diagonal, is 1 (since the representation is unimodular), we see that the isomorphism (51) is canonical.

Definition. One may interpret the composition of isomorphisms (51) and (50) as a nonzero element of the line

$$\rho_K = \rho_{K,M} \in \det(M)^{-\chi(K)} \otimes \det(H_*(K, M)), \quad (53)$$

which will be called *combinatorial L^2 torsion*, or *L^2 Reidemeister - Franz torsion*.

In §5 we will consider analytic version of this construction.

4.5. Remarks.

1. Note that in the case $\mathcal{A} = \mathbb{C}$ we arrive to the classical definitions, cf. [Mi1], [Mi2], [Mi3], [Mu], [BZ].

2. Recall that the classical Reidemeister-Franz torsion is not, in general, a homotopy invariant, so one cannot expect homotopy invariance from our torsion invariant. But the *combinatorial invariance* holds, cf. below.

3. Although our notation ρ_K for the torsion invariant does not involve explicitly the trace $\tau : \mathcal{A} \rightarrow \mathbb{C}$, the whole construction (including the Fuglede-Kadison determinants and the determinant lines) certainly depend of the choice of the trace τ . Thus, in fact, we have one determinant line $\det(M)^{-\chi(K)} \otimes \det(H_*(K, M))$ for each trace τ , forming a bundle, and ρ_K is a section of this bundle.

Consider, for example, the case when the initial von Neumann algebra \mathcal{A} is the group ring of a finite group, $\mathcal{A} = \mathbb{C}[G]$. Then we have one trace for any irreducible representation of G . Thus, we have one determinant line for each irreducible representation, and the torsion ρ_K is a function on the classes of the irreducible representations with values in these determinant lines.

4. In the case when $\chi(K) = 0$, the torsion invariant is just an element of the determinant line of the reduced L^2 -homology $\det(H_*(K, M))$. This is always the case if K is a closed manifold of odd dimension.

5. If the representation M is *unitary*, then the scalar product on M determines a well defined element in $\det(M)$. Then, again, we can consider the torsion as a volume on the reduced L^2 -homology, that is, as an element of $\det(H_*(K, M))$.

Assuming additionally, that the reduced L^2 -homology $H_*(K, M)$ vanishes, we can identify the determinant line $\det(H_*(K, M))$ with \mathbb{R} and so ρ_K is just a number. Under this assumption it was studied in [CM], [Lu], [LR].

4.6. Theorem (Combinatorial Invariance). *Let K be a finite polyhedral cell complex and let K' be its subdivision. Suppose that M is an unimodular representation of $\pi = \pi_1(K)$ over a finite von Neumann algebra \mathcal{A} with a trace τ and suppose the complex $M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K})$ is of determinant class. Let*

$$\psi : H_*(K, M) \rightarrow H_*(K', M) \quad (54)$$

be the isomorphism induced on the reduced L^2 homology by the subdivision chain map. Then ψ is an isomorphism of Hilbertian modules and the induced by ψ map

$$\text{id} \otimes \psi_* : \det(M)^{-\chi(K)} \otimes \det(H_*(K, M)) \rightarrow \det(M)^{-\chi(K)} \otimes \det(H_*(K', M)) \quad (55)$$

maps ρ_K onto $\rho_{K'}$.

Proof. It is enough to consider the elementary subdivision when a single q -dimensional cell e is divided into two q -dimensional cells e_+ and e_- introducing an additional separating $(q-1)$ -dimensional cell e_0 , see the figure.

FIGURE 1

We have the exact sequence of free left $\mathbb{Z}[\pi]$ -chain complexes

$$0 \rightarrow C_*(\tilde{K}) \xrightarrow{\psi} C_*(\tilde{K}') \rightarrow D_* \rightarrow 0 \quad (56)$$

where the chain complex D_* has nontrivial chains only in dimension q and $q-1$ and D_q and D_{q-1} are both free of rank one. The free generator of the module D_q can be labeled with e_+ and the generator of D_{q-1} can be labeled with the cell e_0 and then the boundary homomorphism is given by $\partial(e_+) = e_0$.

Fix an admissible scalar product $\langle \cdot, \cdot \rangle$ on M . Using the cell structures of K and K' , we then obtain canonically the scalar products on the complexes

$$M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K}), \quad M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K}'), \quad M \otimes_{\mathbb{Z}\pi} D_*,$$

and, according to our definitions, these scalar products represent volume forms

$$x \in \det(M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K})), \quad y \in \det(M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K}')), \quad z \in \det(M \otimes_{\mathbb{Z}\pi} D_*)$$

Note that the chain complex D_* is acyclic and so the canonical isomorphism (44), which we will denote in this case by ϕ_{D_*} , identifies the determinant line of D_* with \mathbb{R} . It is easy to see that $\phi_{D_*}(z) = 1$.

Using the exact sequence (56), we obtain the commutative diagram

$$\begin{array}{ccc} \det(M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K}')) & \longrightarrow & \det(M \otimes_{\mathbb{Z}\pi} C_*(\tilde{K})) \otimes \det(M \otimes_{\mathbb{Z}\pi} D_*) \\ \phi_{K'} \downarrow & & \downarrow \phi_K \otimes \phi_{D_*} \\ \det(H_*(K', M)) & \xrightarrow{\psi_*^{-1}} & \det(H_*(K, M)) \end{array}$$

and the upper horizontal map maps y into $x \otimes z$. By the definition, we have

$$\rho_{K'} = \langle \cdot, \cdot \rangle^{-\chi(K)} \otimes \phi_{K'}(y), \quad \text{and} \quad \rho_K = \langle \cdot, \cdot \rangle^{-\chi(K)} \otimes \phi_K(x)$$

and so from the diagram above we get $\text{id} \otimes \psi_*^{-1}(\rho_{K'}) = \rho_K$. \square

4.7. Cohomological version of the construction. Here we will mention a variation of the construction of the combinatorial torsion (53), based on cohomology instead of homology.

Suppose that M is a *Hilbertian* $(\pi - \mathcal{A})$ -bimodule. This notion is similar to the notion of a Hilbertian $(\mathcal{A} - \pi)$ -bimodule, introduced in 4.2. It means that M is a topological vector space supplied with a right \mathcal{A} -action and a left π -action which commute with each other and such that M is a finitely generated Hilbertian module over \mathcal{A} .

If K is a finite polyhedron, consider the cellular chain complex $C_*(\tilde{K})$ of its universal covering \tilde{K} . Then

$$C^*(K, M) = \overline{\text{Hom}_{\mathbb{Z}\pi}(C_*(\tilde{K}), M)}$$

is a cochain complex of finitely generated left Hilbertian modules over \mathcal{A} . Here the bar means that we convert the right \mathcal{A} -module structure on $\text{Hom}_{\mathbb{Z}\pi}(C_*(\tilde{K}), M)$ into a left structure using the involution of \mathcal{A} . The reduced cohomology of $C^*(K, M)$ will be denoted

$$H^*(K, M). \tag{57}$$

We suppose now that *the chain complex $C^*(K, M)$ is of determinant class and that the $(\pi - \mathcal{A})$ -module M is unimodular* (this means that for each $g \in \pi$ the map $M \rightarrow M$, given by multiplication on g from the left has Fuglede-Kadison determinant 1). Then there is a natural isomorphism

$$\det(M)^{\chi(K)} \rightarrow \det(C^*(K, M)),$$

which is similar to (51). Composed with the isomorphism of Proposition 3.10

$$\det(C^*(K, M)) \rightarrow \det(H^*(K, M))$$

(where $H^*(K, M)$ denotes the reduced L^2 cohomology) it gives a canonical element

$$\rho_K = \rho_{K,M} \in \det(M)^{-\chi(K)} \otimes \det(H^*(K, M)). \quad (58)$$

It will be called *combinatorial L^2 torsion*, or *L^2 Reidemeister - Franz torsion*.

Again, similar to 4.5, remark 5, we may view the torsion ρ_K as an element of the line $\det(H^*(K, M))$, assuming that the representation π is unitary and so M has a specified admissible invariant scalar product.

§5. Analytic L^2 -torsion

In this section we generalize the classical construction of analytic torsion of D.B. Ray and I.M. Singer [RS] to the case of infinite dimensional representations of the fundamental group. The invariant we construct represents a volume form on the reduced L^2 cohomology. We will see in the next section that this allows really understand the meaning of the recent theorem of Burghelea, Friedlander, Kappeler and McDonald [BFKM].

Throughout this section \mathcal{A} denotes a finite von Neumann algebra with a fixed finite, normal, and faithful trace τ .

5.1 Flat Hilbertian \mathcal{A} -Bundles. Let M be a finitely generated Hilbertian $(\pi - \mathcal{A})$ -bimodule (cf. 4.2). Recall that this means that \mathcal{A} acts on M from the right and M , with respect to this action, is a finitely generated Hilbertian module (cf. 1.3); also, π is a discrete group acting on M from the left and the action of π commutes with that of \mathcal{A} .

Let X be a connected, closed, smooth manifold with fundamental group π . A *flat Hilbertian \mathcal{A} -bundle with fiber M over X* is an associated bundle $p : \mathcal{E} \rightarrow X$, where $\mathcal{E} = (M \times \tilde{X}) / \sim$ with its natural projection onto X . Here $(v, x) \sim (gv, gx)$ for all $g \in \pi$, $x \in \tilde{X}$ and $v \in M$. Here \tilde{X} denotes the universal covering of X . The map $p : \mathcal{E} \rightarrow X$ is then a locally trivial bundle of topological vector spaces (cf. chapter 3 of [La]), which has a natural fiberwise left action of \mathcal{A} .

Any smooth section s of $\mathcal{E} \rightarrow X$ can be uniquely represented by a smooth equivariant map $\phi : \tilde{X} \rightarrow M$, that is, $\phi(gx) = g\phi(x)$ for all $g \in \pi$ and $x \in \tilde{X}$.

5.2. Given a flat Hilbertian \mathcal{A} -bundle $\mathcal{E} \rightarrow X$ over a closed connected manifold X , one can consider the space of smooth differential j -forms on X with values in \mathcal{E} . This space will be denoted by $\Omega^j(X, \mathcal{E})$. It is naturally defined as a left \mathcal{A} -module. An element of $\Omega^j(X, \mathcal{E})$ can be uniquely represented as a π -invariant differential form in $M \otimes_{\mathbb{C}} \Omega^j(\tilde{X})$. Here one considers the total (diagonal) π action, that is, the tensor product of the actions of π on $\Omega^j(\tilde{X})$ and on M . More precisely, if $\omega \in \Omega^j(\tilde{X})$ and $v \in M$, then $v \otimes \omega$ is said to be π -invariant if $vg^{-1} \otimes g^*\omega = v \otimes \omega$ for all $g \in \pi$.

A *flat \mathcal{A} -linear connection* on a flat Hilbertian \mathcal{A} -bundle \mathcal{E} is defined as an \mathcal{A} -homomorphism

$$\nabla : \Omega^j(X, \mathcal{E}) \rightarrow \Omega^{j+1}(X, \mathcal{E})$$

such that

$$\nabla(f\omega) = df \wedge \omega + f\nabla(\omega) \quad \text{and} \quad \nabla^2 = 0$$

for any \mathcal{A} valued function f on X and for any $\omega \in \Omega^j(X, \mathcal{E})$. On the flat Hilbertian \mathcal{A} -bundle \mathcal{E} , as defined in the previous paragraph, there is a *canonical flat \mathcal{A} -linear*

connection ∇ , which is given as follows: under the identification of $\Omega^j(X, \mathcal{E})$ given in the previous paragraph, one defines $\nabla(v \otimes \omega) = v \otimes d\omega$, where d is the De Rham exterior derivative.

5.3 Hermitian metrics and L^2 scalar products. A Hermitian metric h on a flat Hilbertian \mathcal{A} -bundle $p : \mathcal{E} \rightarrow X$ is a smooth family of admissible scalar products on the fibers. Any Hermitian metric on $p : \mathcal{E} \rightarrow X$ defines a wedge product

$$\wedge : \Omega^i(X, \mathcal{E}) \otimes \Omega^j(X, \mathcal{E}) \rightarrow \Omega^{i+j}(X)$$

similar to the finite dimensional case.

A Hermitian metric on $p : \mathcal{E} \rightarrow X$ together with a Riemannian metric on X determine an scalar product on $\Omega^i(X, \mathcal{E})$ in the standard way; namely, using the Hodge star operator

$$* : \Omega^j(X, \mathcal{E}) \rightarrow \Omega^{n-j}(X, \mathcal{E})$$

one sets

$$(\omega, \omega') = \int_X \omega \wedge * \omega'.$$

With this scalar product $\Omega^i(X, \mathcal{E})$ becomes a pre-Hilbert space. Define the space of L^2 differential j -forms on X with coefficients in \mathcal{E} , which is denoted by $\Omega_{(2)}^j(X, \mathcal{E})$, to be the Hilbert space completion of $\Omega^j(X, \mathcal{E})$.

5.4 Reduced L^2 cohomology. Given a flat Hilbertian \mathcal{A} bundle $p : \mathcal{E} \rightarrow X$, one defines the *reduced L^2 cohomology of \mathcal{E}* as the quotient

$$H^j(X, \mathcal{E}) = \frac{\ker \nabla / \Omega_{(2)}^j(X, \mathcal{E})}{\text{cl}(\text{im } \nabla / \Omega_{(2)}^{j-1}(X, \mathcal{E}))}, \quad (59)$$

where the connection ∇ on \mathcal{E} is assumed to be extended to an unbounded, densely defined operator $\Omega_{(2)}^j(X, \mathcal{E}) \rightarrow \Omega_{(2)}^{j+1}(X, \mathcal{E})$. Then $H^j(X, \mathcal{E})$ is naturally defined as a Hilbertian module over \mathcal{A} . Considered as Hilbertian module over \mathcal{A} , it is independent on the choice of the Riemannian metric on X and the Hermitian metric on \mathcal{E} .

The corresponding L^2 Betti numbers are denoted by

$$b^j(X, \mathcal{E}) = \dim_{\tau} (H^j(X, \mathcal{E})). \quad (60)$$

Given a smooth triangulation of the manifold X , one may consider also the reduced L^2 -cohomology defined combinatorially (57) with coefficients in the monodromy representation M of \mathcal{E} . The De Rham type theorem states that these two kinds of reduced L^2 cohomology $H^*(X, M)$ (the combinatorial (57)) and $H^*(X, \mathcal{E})$ (the analytic (59)) are *canonically isomorphic viewed as Hilbertian \mathcal{A} -modules*. The canonical isomorphism

$$I : H^*(X, \mathcal{E}) \rightarrow H^*(X, M) \quad (61)$$

is given by integration of forms along the simplices. The isomorphism (61) was established by J. Dodziuk in [D]. In fact, Dodziuk studied only the case of the regular representation but his arguments can be easily generalized to the case of an arbitrary flat bundle.

5.5. Let $\Delta_j = \int_0^\infty \lambda dE_j(\lambda)$ denote the spectral decomposition of the Laplacian $\Delta_j = \nabla \nabla^* + \nabla^* \nabla : \Omega_{(2)}^j(X, \mathcal{E}) \rightarrow \Omega_{(2)}^j(X, \mathcal{E})$, where ∇^* denotes the formal adjoint of ∇ with respect to the L^2 scalar product on $\Omega_{(2)}^j(X, \mathcal{E})$. Note that by definition, the Laplacian is a formally self-adjoint operator, which is densely defined. We also denote by Δ_j the self adjoint extension of the Laplacian.

Recall, that the *spectral density function* is defined as $N_j(\lambda) = \text{Tr}_\tau(E_j(\lambda))$ and the *theta function* is by definition

$$\theta_j(t) = \int_0^\infty e^{-t\lambda} dN_j(\lambda) = \text{Tr}_\tau(e^{-t\Delta_j}) - b^j(X, \mathcal{E}).$$

Here we use the well known fact that the projection $E_j(\lambda)$ and the heat operator $e^{-t\Delta_j}$ have smooth Schwartz kernels, which are smooth sections of a bundle over $X \times X$ with fiber the commutant of M , cf. [BFKM], [GS], [Luk]. The symbol Tr_τ denotes application of the canonical trace on the commutant (cf. Proposition 1.8) to the restriction of the kernels to the diagonal followed by integration over the manifold X . See also [M], [L] and [GS] for the case of the flat bundle defined by the regular representation of the fundamental group.

5.6 Definition. A flat Hilbertian \mathcal{A} -bundle $\mathcal{E} \rightarrow X$ is said to be of *determinant class* if

$$\int_0^1 \log(\lambda) dN_j(\lambda) > -\infty \quad \text{or, equivalently} \quad \int_1^\infty t^{-1} \theta_j(t) dt < \infty. \quad (62)$$

for all $j = 0, \dots, n$. Equivalence of these two conditions was proved in [BFKM], Proposition 2.12.

This notion has been studied earlier. In [CM], this property was called *D-acyclicity*. In [BFKM], the term *a-determinant class* was used. In [BFKM], D. Burghelea et al. introduced also the notion of *flat bundles of c-determinant class* (here "a" stands for *analytic* and "c" stands for *combinatorial*); these two notions are actually equivalent as shown by A.V.Efremov [E]. The c-determinant class condition is obviously equivalent to Definition 3.8. This shows in particular that the *determinant class property of a flat Hilbertian \mathcal{A} bundle does not depend on the choice of metrics g on X and h on the bundle \mathcal{E} .*

5.7. Definition ([BFKM]). For $\lambda > 0$, the *zeta function of the Laplacian Δ_j* is defined on the half-plane $\Re(s) > n/2$ as

$$\zeta_j(s, \lambda, \mathcal{E}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} \theta_j(t) dt. \quad (63)$$

It has been shown by [BKFM], that $\zeta_j(s, \lambda, \mathcal{E})$, viewed as a function of s , is holomorphic in the half-plane $\Re(s) > n/2$ (where $n = \dim X$) and has a meromorphic continuation to \mathbb{C} with no pole at $s = 0$. Under the hypothesis that the flat Hilbertian \mathcal{A} -bundle $\mathcal{E} \rightarrow X$ is of determinant class, the limit $\lim_{\lambda \rightarrow 0} \zeta_j'(0, \lambda, \mathcal{E})$ exists, where the prime denotes the differentiation with respect to s ; we will denote this limit by $\zeta_j'(0, 0, \mathcal{E})$. Denote also

$$\zeta'(0, 0, \mathcal{E}) = \sum_j (-1)^j j \zeta_j'(0, 0, \mathcal{E}). \quad (64)$$

5.8. The construction of L^2 analytic torsion. Let (X, g) be a closed Riemannian manifold of dimension n with $\pi = \pi_1(X)$ and let \mathcal{E} be a flat Hilbertian \mathcal{A} -bundle over X with a Hermitian metric h and with fibre M . We assume that \mathcal{E} is of determinant class.

Using the Hodge theorem (cf. [D], [BFKM], [GS]) one obtains an identification between the reduced L^2 cohomology $H^j(X, \mathcal{E})$ (defined by (59)) and the space of harmonic forms, i.e. the kernel of the Laplacian Δ_j acting on $\Omega_{(2)}^j(X, \mathcal{E})$. Since the last space is embedded into a Hilbert space it inherits an admissible scalar product and thus we obtain an admissible scalar product on the cohomology space $H^j(X, \mathcal{E})$.

These admissible scalar products on $H^j(X, \mathcal{E})$, determine nonzero elements of the determinant lines $\det(H^j(X, \mathcal{E}))$ for all j and thus, their alternating product in the line

$$\det(H^*(X, \mathcal{E})) = \prod_{j=0}^n \det(H^j(X, \mathcal{E}))^{(-1)^j} \quad (65)$$

is defined; the last element we will denote by $\tilde{\rho}(g, h)$. This notation emphasizes dependence on the metrics g and h .

Definition. The L^2 analytic torsion is defined as the element of the determinant line $\rho_{\mathcal{E}}(g, h) \in \det(H^*(X, \mathcal{E}))$, where

$$\rho_{\mathcal{E}}(g, h) = e^{\frac{1}{2}\zeta'(0,0,\mathcal{E})} \cdot \tilde{\rho}(g, h). \quad (66)$$

Here $\zeta'(0,0,\mathcal{E})$ is defined in (64). Observe that the sign in the exponent of (66) is different from the one in the usual formulas (as in [BZ], [BGS]); it happens because we consider the elements of the corresponding lines (the volume forms) but not the metrics on them.

Thus, the L^2 analytic torsion represents a volume form on the reduced L^2 cohomology. We will see later in section 6, that $\rho_{\mathcal{E}}(g, h)$ does not depend on the metrics g and h if $\dim(X)$ is odd.

Note, that in the case when $\mathcal{A} = \mathbb{C}$, we arrive at the classical definition of the Ray-Singer-Quillen metric on the determinant of the cohomology.

Assuming that the reduced L^2 cohomology $H^*(X, \mathcal{E})$ vanishes, we can identify canonically the determinant line $\det(H^*(X, \mathcal{E}))$ with \mathbb{R} , and so the torsion $\rho_{\mathcal{E}}$ just defined turns into a number. In this case, it coincides with the invariant studied in [M] and [L].

§6. Theorem of Burghelea, Friedlander, Kappeler and McDonald

In this section we will show that a recent theorem of Burghelea, Friedlander, Kappeler and McDonald [BFKM] can be naturally formulated in terms of the notions introduced in the present paper. Namely, we will show that the main theorem of [BFKM] (Theorem 2) is equivalent to the statement that the L^2 combinatorial and L^2 analytic torsion invariants, as defined in the present paper, determine identical volume forms on the reduced L^2 cohomology (cf. Theorem 6.1 below for the precise statement). Note that the original theorem of Burghelea et al. [BFKM] is formulated using three numerical torsion type invariants, neither of which is a topological invariant.

We are very thankful to the referee, who pointed out this statement to us.

6.1. Theorem. *Let X be a closed odd-dimensional Riemannian manifold and let $\mathcal{E} \rightarrow X$ be a flat Hilbertian bundle (cf. 5.1) of determinant class supplied with a flat fiberwise Hermitian metric h . Consider the reduced L^2 cohomology $H^*(X, \mathcal{E})$ and the analytic L^2 torsion $\rho_{\mathcal{E}}(g, h) \in \det(H^*(X, \mathcal{E}))$. On the other hand, consider a smooth triangulation of X and the combinatorially defined reduced L^2 cohomology $H^*(X, M)$ (cf. (57), where M denotes the fiber of \mathcal{E} , viewed together with the monodromy representation of the fundamental group of X and with the induced admissible scalar product. Then the L^2 combinatorial torsion $\rho_X \in \det(H^*(X, M))$ is defined, cf. (58). Then the isomorphism between the cohomological determinant lines $I_* : \det(H^*(X, \mathcal{E})) \rightarrow \det(H^*(X, M))$ determined by the De Rham isomorphism (61) (cf. 2.3) maps the analytically defined torsion $\rho_{\mathcal{E}}(g, h)$ onto the combinatorially defined torsion ρ_X , i.e.*

$$I_*(\rho_{\mathcal{E}}(g, h)) = \rho_X. \quad (67)$$

In particular we obtain that the analytic torsion $\rho_{\mathcal{E}}(g, h)$ does not depend on the Riemannian metric g on X and Hermitian metric h on \mathcal{E} .

Proof. The proof consists of interpreting the result of [BFKM] in terms of the notions of this paper. Let us denote by $\alpha \in \det(H^*(X, M))$ the volume form on the combinatorially defined reduced L^2 cohomology corresponding to the given triangulation (i.e. corresponding to the metric on the combinatorial harmonic forms via the Hodge isomorphism). Denote by $\beta \in \det(H^*(X, \mathcal{E}))$ the volume form on the analytically defined reduced L^2 cohomology which corresponds to the induced admissible scalar product on the space of harmonic forms via the Hodge isomorphism.

We claim that

$$I_*(\beta) = T_{met}^{-1} \cdot \alpha, \quad (68)$$

where T_{met} (the torsion of the metric) is a numerical invariant, measuring the relative size of the Riemannian metric and the triangulation, and is defined in the introduction to [BFKM]. Formula (68) follows by comparing the definition of [BFKM] with our definitions: formulae (26) and (23).

Now, again by comparing the definitions of [BFKM] with ours, we obtain for the combinatorial torsion

$$\rho_X = T_{comb} \cdot \alpha, \quad (69)$$

and also

$$\rho_{\mathcal{E}}(g, h) = T_{an} \cdot \beta \quad (70)$$

for the analytic torsion, where T_{comb} and T_{an} are the numbers introduced in [BFKM].

Therefore, combining (68), (69) and (70), we obtain

$$\begin{aligned} I_*(\rho_{\mathcal{E}}(g, h)) &= T_{an} \cdot I_*(\beta) = \\ &= T_{an} \cdot T_{met}^{-1} \cdot \alpha = \\ &= T_{comb} \cdot \alpha = \rho_X. \end{aligned}$$

This completes the proof. \square

Note that Theorem 6.1 implies Theorem 4.6 in the case when the polyhedron is a manifold of odd dimension and the bundle is unitary.

It is plausible that Theorem 6.1 holds without assuming that the flat bundle $\mathcal{E} \rightarrow X$ is unitary. In fact, the analytic torsion (66) is defined without assumptions of this kind. One may show that $\rho(g, h)$ is independent of the metrics g and h , if the dimension of X is odd (this is a general property of analytic torsion for elliptic complexes, cf. [F3]). In order to be able to remove the assumption of unitarity (or unimodularity) from Theorem 6.1 one has to be able first to define the combinatorial torsion (58) without assuming unimodularity of the bundle $\mathcal{E} \rightarrow X$. It is well known that it is impossible for polyhedra; but it is possible, if one restricts to orientable manifolds X of odd dimension. A purely combinatorial construction (using Poincaré duality) was suggested in [F] for finite dimensional flat bundles; the corresponding metric on the determinant line of the cohomology was called *Poincaré - Reidemeister metric*. We conjecture that the construction of the Poincaré - Reidemeister metric of [F] can be generalized to flat Hilbertian bundles of determinant class and that Theorem 6.1 can then be generalized as stating that the Poincaré - Reidemeister norm (defined combinatorially) of the analytic torsion (66) equals 1. Compare [F], where a similar theorem was proven (using theorem of Bismut and Zhang [BZ]) in the finite dimensional case.

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